Direct Component Modal Synthesis Technique for System Dynamic Analysis

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A method for substructure modal synthesis to determine the modal parameters of a structure is investigated. The order of the solution matrix does not depend on the number of substructure modes used. Instead, it depends only on the number of connections between substructures. The eigensolution determined by this method is exact. Since the system dynamic matrix equation is not needed in this method, numerical computations required for the solution are drastically reduced. The eigensolutions of the system in any frequency range can be obtained independently. Computation is reduced because lower or higher eigensolutions outside the interested frequency range are not required to be calculated. The results obtained by this technique and the analytical method are compared, and they are identical.

Nomenclature

f = force vector in frequency domain
$\bar{f}$ = modal force vector from $\bar{H}\bar{f}=0$
f_{IJ} = force vector common to substructures I and J
$f(t)$ = force vector in time domain
$\bar{H}(\omega)$ = modal force matrix from $\bar{H}\bar{f}=0$
$H^I_{ij}(\omega)$ = frequency transfer matrix of substructure I at row i, column j
K = stiffness matrix
M = mass matrix
N_I = number of modes in substructure I
$p(t)$ = modal coordinate vector
x = response vector in frequency domain
$x(t)$ = response vector in time domain
x^I = subvector of system eigenvector belongs to substructure I
x_{IJ} = response vector common to substructures I and J
Λ = diagonal matrix of eigenvalues
λ_r = rth eigenvalue of a structure
Φ = $[\phi_1\ \phi_2 \ldots \phi_n]$ modal matrix
ϕ_r = rth eigenvector or modal vector
ϕ^I_r = rth eigenvector of substructure I
ω = frequency of excitation
ω_r = rth natural frequency of the system
$(\ldots)^I$ = superscript to indicate substructure I

Introduction

MANY modal synthesis methods have been presented in the past, such as those of Hurty,[1] Hale and Meirovitch,[2] Craig and Chang,[3] and Hou.[4] Some used the substructure modes, geometric compatibility, and force equilibrium equations to assemble the system matrix. If too many substructure modes are used, the system matrix of the structure can be large, and so will the number of solution calculations. Benfield and Hruda,[5] Rubin,[6] and Hintz[7] investigated the component modal synthesis by approximations or transformations. With the approximation method, the order of the system matrix is reduced, but some accuracy of the solution is compromised. Guyan[8] proposed the method for reduction of stiffness and mass matrices. Geering[9] used the projection matrix to reduce the order of the system matrix to the relevant degrees of freedom. That method requires many matrix inversions and fails at the frequencies equal to the substructure natural frequencies. Kuang and Tsuei[10] introduced another approach. Different sets of substructure modes were selected and used to determine the eigensolution of the structure in different frequency ranges. For a wide frequency range, this method is not effective. Kubomura[11] used the transformation method to reduce the number of degrees of freedom in the system. However, the higher and lower frequency modes of the substructure were approximated. The present method for substructure modal synthesis without the aforementioned limitations is proposed here for determining the exact modal parameters of the synthesized structure.

Received June 15, 1987; revision received March 6, 1988.

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Modal Analysis for Substructure

Systems can be divided into substructures. Each substructure has its own equation of motion and can be connected to other substructures through joints.

The equation of motion for a substructure is

$$M\ddot{x}(t) + Kx(t) = f(t) \tag{1}$$

The natural frequencies and mode shapes of the substructure can be obtained by solving the differential equation

$$M\ddot{x}(t) + Kx(t) = 0 \tag{2}$$

Let λ_r and ϕ_r be the rth mode eigenvalue and eigenvector. The following orthogonality properties

$$\Phi^T M \Phi = I, \qquad \Phi^T K \Phi = \Lambda$$

are obtained, where Λ is a diagonal matrix and the modal matrix Φ is defined by

$$\Phi = [\phi_1\ \phi_2 \ldots \phi_n]$$

Since Φ is the modal matrix obtained by solving Eq. (2), the displacement vector $x(t)$ can be represented by

$$x(t) = \Phi p(t)$$

where $p(t)$ is the modal coordinate vector of the substructure.

By using the preceding transformation, Eq. (1) becomes

$$M\Phi\ddot{p}(t) + K\Phi p(t) = f(t)$$

For harmonic excitation at a frequency ω, which is different from the natural frequency λ, the response function can be expressed as

$$x(t) = \Phi[\Lambda - \omega^2 I]^{-1}\Phi^T f(t) \qquad \text{or} \qquad x = H(\omega)f \qquad (3)$$

where $f(t) = fe^{i\omega t}$ and $x(t) = xe^{i\omega t}$ are used. It is noted that the vector x is a function of frequency. Since $[\Lambda - \omega^2 I]$ is a diagonal matrix, the transfer matrix $H(\omega) = \Phi[\Lambda - \omega^2 I]^{-1}\Phi^T$ can be obtained by simple multiplication when the eigensolution Φ of the substructures is provided.

Selection of Substructure Modes for Synthesis and Partition of Transfer Matrix $H(w)$

Because of the special nature of Eq. (3), the complete substructure modal set is not needed to analyze the system dynamics at a given frequency range. For example, if the $H(w)$ of a substructure at a frequency range $w_b < w < w_c$ is desired, the substructure modes below w_a and above w_d, where $w_a \ll w_b$ and $w_d \gg w_c$, can be excluded. The contribution of the modes above w_d and below w_a have negligible effects on the transfer function matrix $H(w)$ at the given frequency range $w_b < w < w_c$. Therefore, only the substructure modes within the frequency range $w_a < w < w_d$ are to be included in analyzing the system dynamics. The substructure modes at this particular desired frequency range for a complicated substructure are usually obtained by finite-element analysis or experimental testing.

Just like the other modal synthesis methods, there is no universal rule of how to select the lower and upper frequency bounds (w_a and w_d) of the substructure modes to be included in the analysis. The frequency bounds selected depend on the degree of accuracy the engineer requires for the results of the analysis.

Equation (3) for a substructure I can be partitioned as

$$\begin{bmatrix} x_{II} \\ x_{JI} \\ \cdots \\ \cdots \\ \cdots \\ x_{IK} \end{bmatrix} = \begin{bmatrix} H^I_{II}(\omega) & H^I_{IJ}(\omega) & \cdots & H^I_{IK}(\omega) \\ H^I_{JI}(\omega) & H^I_{JJ}(\omega) & \cdots & H^I_{JK}(\omega) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ H^I_{KI}(\omega) & H^I_{KJ}(\omega) & \cdots & H^I_{KK}(\omega) \end{bmatrix} \begin{bmatrix} f_{II} \\ f_{IJ} \\ \cdots \\ \cdots \\ \cdots \\ f_{IK} \end{bmatrix} \qquad (4)$$

where x_{IJ} is the response (displacement) vector and f_{IJ} is the force vector, common to substructures I and J, and the superscript I refers to substructure I.

General Discussion of the Dynamic Equation of a System

Substructure modes, geometric compatibility, and force equilibrium are used to set up the system dynamic equation. The dynamic equation of the system is represented by

$$\hat{M}\ddot{q} + \hat{K}q = 0 \qquad (5)$$

If a system has s substructures and substructure I has N_I substructure modes, the total number N_t of substructure modes used to formulate Eq. (5) is

$$N_t = \sum_{I=1}^{s} N_I$$

For a system with N_t substructure modes and r constraint equations, the final dynamic equation of the system is in the order of $N_t - r$. A complex system may have many substructure modes, and to obtain the system eigensolution from Eq. (5) requires many numerical computations.

Modal Force Equation for the System Eigensolutions

The approach suggested here does not require setting up the dynamic equation of the system. Instead, the system eigensolution is obtained from the Modal Force equation, which will be defined in Eq. (9). A system with three substructures, as shown in Fig. 1, is used to illustrate the synthesis method. Substructures 1 and 3 are connected to substructure 2, but not to each other. For natural vibrations, the external forces acting on the system are zero.

From Eq. (4), the response equations for substructures 1, 2, and 3, respectively, become

$$x^1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} H^1_{11}(\omega) & H^1_{12}(\omega) \\ H^1_{21}(\omega) & H^1_{22}(\omega) \end{bmatrix} \begin{bmatrix} 0 \\ f_{12} \end{bmatrix} \qquad (6a)$$

$$x^2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} H^2_{11}(\omega) & H^2_{12}(\omega) & H^2_{13}(\omega) \\ H^2_{21}(\omega) & H^2_{22}(\omega) & H^2_{23}(\omega) \\ H^2_{31}(\omega) & H^2_{32}(\omega) & H^2_{33}(\omega) \end{bmatrix} \begin{bmatrix} f_{21} \\ 0 \\ f_{23} \end{bmatrix} \qquad (6b)$$

$$x^3 = \begin{bmatrix} x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} H^3_{11}(\omega) & H^3_{12}(\omega) \\ H^3_{21}(\omega) & H^3_{22}(\omega) \end{bmatrix} \begin{bmatrix} f_{32} \\ 0 \end{bmatrix} \qquad (6c)$$

The conditions for geometric compatibility and force equilibrium are expressed as

$$x_{IJ} = x_{JI}, \qquad f_{IJ} = -f_{JI} \qquad (7)$$

System Synthesis and Modal Force Matrix

By retaining only the joint coordinates, the following expressions can be obtained from Eqs. (6):

$$x_{12} = H^1_{22}(\omega)\, f_{12} \qquad (8a)$$

$$x_{21} = H^2_{11}(\omega)\, f_{21} + H^2_{13}(\omega)\, f_{23} \qquad (8b)$$

$$x_{23} = H^2_{31}(\omega)\, f_{21} + H^2_{33}(\omega)\, f_{23} \qquad (8c)$$

$$x_{32} = H^3_{11}(\omega)\, f_{32} \qquad (8d)$$

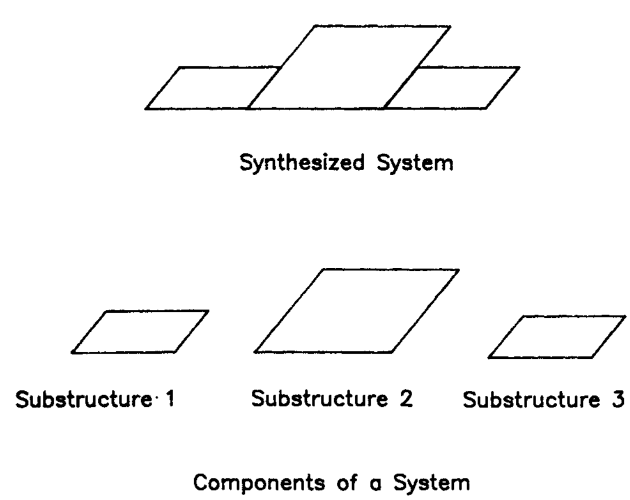

Fig. 1 Synthesized system and its three substructures.

Applying the conditions of compatibility and equilibrium at the joints of the substructures, the following equation is obtained:

$$\begin{bmatrix} H_{22}^1(\omega) + H_{11}^2(\omega) & -H_{13}^2(\omega) \\ -H_{31}^2(\omega) & H_{33}^2(\omega) + H_{11}^3(\omega) \end{bmatrix} \begin{bmatrix} f_{12} \\ f_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9a}$$

or

$$\bar{H}\bar{f} = 0 \tag{9b}$$

where $\bar{H}$ is defined as the Modal Force matrix and $\bar{f}$ as the Modal Force vector.

Equations (9) are called the modal force equations, which is also the most important equation in the method. It is emphasized that Eqs. (9) satisfy the compatibility, equilibrium, and substructure response equations in the system. The order of the modal force equation depends only on the joints of the substructure. As mentioned before, the transfer matrix $H(\omega)$ is a function of frequency and is obtained by simple multiplication. Since no inversions of large matrices are involved in determining the eigensolutions of the system, the numerical computation is greatly reduced compared to that of the system dynamic equation. Furthermore, the transfer matrix $H(\omega)$ can be stored for use in other system configurations. In short, the system eigensolutions can be derived from the modal force equation, Eq. (9), instead of from the system dynamic equation, Eq. (5).

Natural Frequency of the System

The natural frequencies of the system are determined from the determinant of the modal force matrix $\bar{H}(\omega)$:

$$\det \begin{vmatrix} H_{22}^1(\omega) + H_{11}^2(\omega) & -H_{13}^2(\omega) \\ -H_{31}^2(\omega) & H_{33}^2(\omega) + H_{11}^3(\omega) \end{vmatrix} = 0 \tag{10}$$

Multiplying Eq. (10) by

$$\prod_{I=1}^{s} \prod_{j=1}^{N_I} (\lambda_j^{(I)} - \omega^2)$$

the following polynomial equation is obtained in the form of

$$\sum_{j=0}^{n} a_j \omega^{2j} = 0 \tag{11}$$

where s is the number of substructures and $\lambda_j^{(I)}$ is the jth eigenvalue of the Ith substructure. The n is the total number of the system normal modes, and a_j are the coefficients of the polynomial. It should be emphasized that the system natural frequency equation is valid in any frequency range. In particular, it is also capable of determining the system natural frequencies, which are identical to the substructure natural frequencies.

Modal Force Vector

After the value ω_r is calculated from the polynomial equation, the eigenvector $\bar{f}$ for the rth mode can be obtained from the modal force equation, Eq. 9:

$$\bar{f} = \begin{bmatrix} f_{12} \\ f_{23} \end{bmatrix}$$

Mode Shape

The rth mode system eigenvector x is recovered by substituting the corresponding modal force vector $\bar{f}$ back into Eq. (6). In case a natural frequency of the system is identical to that of the substructure, Eq. (4) cannot be used, and a modified approach should be considered. Discussion of the modified method is presented in the Appendix.

Main Differences Between Other Modal Synthesis Techniques and the Proposed Modal Force Method

This section discusses the differences between the other modal synthesis methods and the proposed method. Most of the modal synthesis methods mentioned in the introduction of this paper formulate the system dynamic equation, Eq. (5), to obtain the system eigensolution. If there are a total of 120 substructure modes to be included in the synthesis and there are 10 physical connections between substructures, the order of the system dynamic equation will be 110 (the number of substructure modes minus the number of physical connections), as previously discussed.

The proposed modal force method uses the modal force matrix equation, Eq. (9), to calculate the system eigensolution. With the same number of substructure modes (120) and physical connections (10) between substructures, the order of the modal force matrix equation is 10 (the number of physical connections between substructures).

The difference between the modal force method and the other methods is apparent at this point. The higher the number of substructure modes included in the analysis, the higher the order for which the system dynamic equation, Eq. 5, has to be solved for the other modal synthesis methods. However, this is not the case for the modal force method. The order of the modal force matrix, Eq. 9, remain constant and is equal to the number of physical connections between substructures. Thus, as the total number of substructure modes increases, the modal force method becomes more efficient when it is compared to the other methods.

Although many frequencies of $\bar{H}(\omega)$ have to be computed, the amount of calculations is still less than the calculations required to solve the larger order of the system dynamic equation, Eq. (5). There are other advantages of this method, and they are summarized and discussed in the conclusion of this paper.

Interpretation of the Synthesis Method

To better understand the previously discussed synthesis procedures, a graphical interpretation is presented here. In order to facilitate the illustration, the x_{12}, x_{21}, x_{23}, x_{32} in Eq. (8) are assumed to be one-dimensional vectors. In this particular case, they are scalar functions of frequency and are represented by curves 1, 2, 3, and 4, respectively, in Fig. 2.

If the intersection of the curves of x_{12}, x_{21} and the intersection of the curves of x_{23}, x_{32} occur at the same ω, then this particular ω is one of the natural frequencies of the system, as in the case of the ω_c, shown in Fig. 2. Since curves 1 and 2 intersect at ω_a but curves 3 and 4 do not, ω_a is not the natural frequency of the system. Similarly, this is the case for ω_b.

In Fig. 2, the modal force equation can be interpreted as two lines L_1 and L_2 in the f_{12} and f_{23} space such that when ω varies, lines L_1 and L_2 rotate about the origin. If L_1 and L_2 coincide at any frequency ω, then that particular ω is one of the natural frequencies of the synthesized system, and the corresponding relation of f_{12} and f_{23} gives the modal force vector $\bar{f}$ of that mode. When the eigenvalues are distinct, the number of ω at which L_1 and L_2 coincide is the same as the number of the degrees of freedom of the synthesized system.

Examples

Example 1

An antenna structure consisting of a thin-wall cylinder[12] and a beam is analyzed. The antenna structure and its subsystems are shown in Fig. 3. For the cylinder structure, nine normal modes are used for the analysis. These nine modes include three rigid-body modes, four prismatic modes, and two nonprismatic modes. For the beam structure, three rigid-body modes and six flexible modes are used for calculation. The resulting modal force matrix equation is of order 3. The normalized determinant curve of the modal force matrix of this system is shown in Fig. 4. The first three system natural fre-

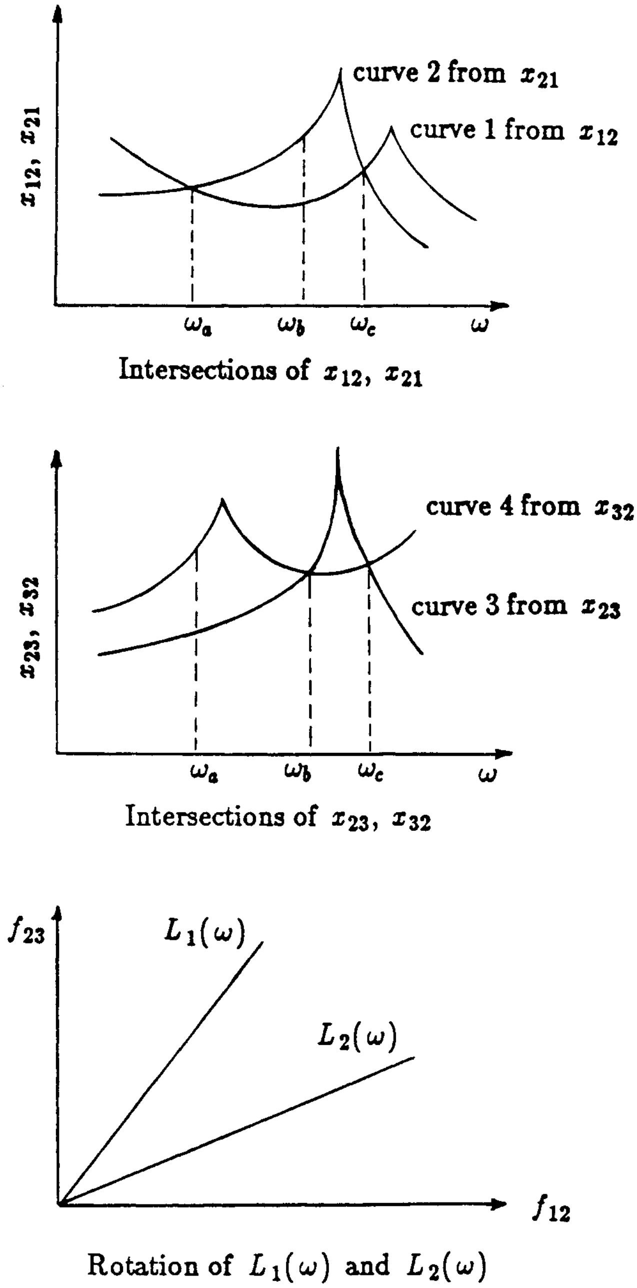

Fig. 2 Determination of the natural frequency of a synthesized system.

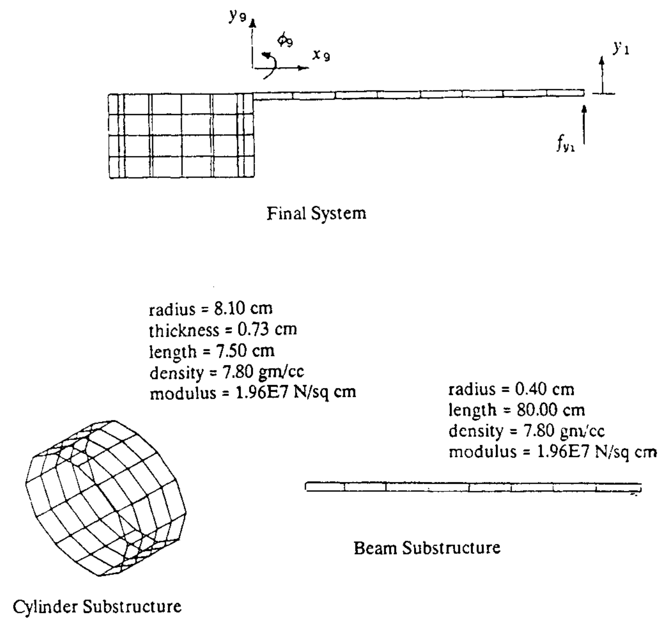

Fig. 3 Antenna structure and its components.

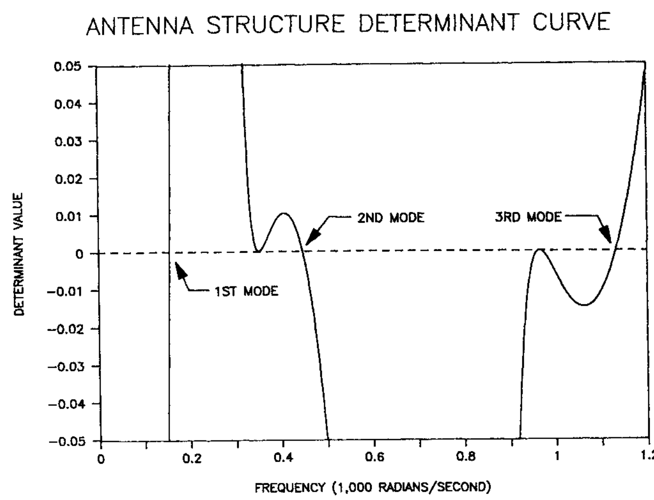

Fig. 4 Normalized determinant value of the antenna structure vs frequency.

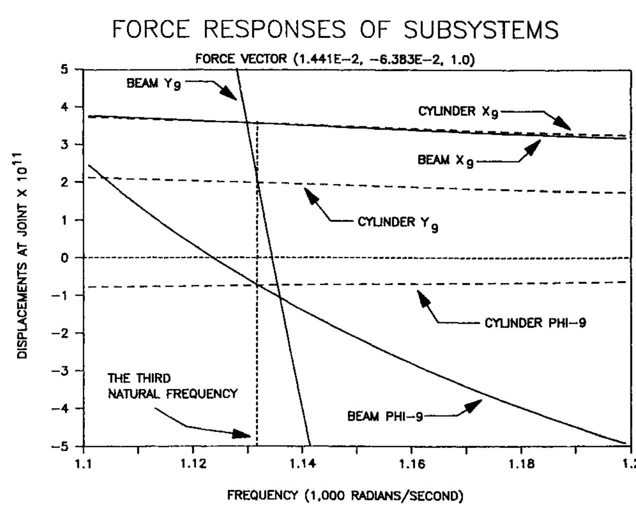

Fig. 5 Force responses of the antenna substructures due to the excitation of the third Modal Force vector.

quencies are found to be 156, 447, and 1132 rad/s for modes 1, 2, and 3, respectively. To further demonstrate the modal force method, the third modal force vector is applied to the subsystems. The displacements at the joint location between substructures are computed and plotted in Fig. 5. It can be seen from Fig. 5 that the displacements x, y, and θ_z of one substructure intersect with the corresponding displacements of the other substructure simultaneously at the third natural frequency.

Example 2

Transverse vibration of a two-beam structure, shown in Fig. 6, is analyzed. Substructure 1 is clamped on the left end and free on the right end. Substructure 2 is free on the left end but clamped on the right end.

The response of substructure 1 is

$$\begin{bmatrix} y^1(x) \\ \theta^1(x) \end{bmatrix} = \begin{bmatrix} a_{11}(x,L) & a_{12}(x,L) \\ a_{21}(x,L) & a_{22}(x,L) \end{bmatrix} \begin{bmatrix} \hat{f}_1 \\ \hat{m}_1 \end{bmatrix} \tag{12a}$$

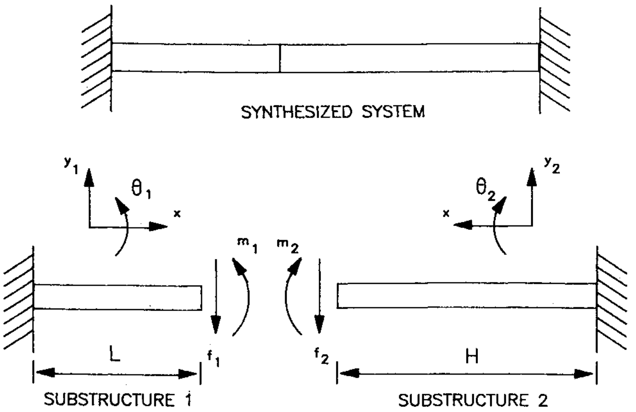

Fig. 6 Transverse vibration of a clamped-clamped beam by synthesis.

The response of substructure 2 is

$$\begin{bmatrix} y^2(z) \\ \theta^2(z) \end{bmatrix} = \begin{bmatrix} a_{11}(z,H) & a_{12}(z,H) \\ a_{21}(z,H) & a_{22}(z,H) \end{bmatrix} \begin{bmatrix} \hat{f}_2 \\ \hat{m}_2 \end{bmatrix} \tag{12b}$$

where

$$a_{11}(x,L) = (\sinh\beta x - \sin\beta x)(\cos\beta L + \cosh\beta L)d_L + (\cos\beta x - \cosh\beta x)(\sin\beta L + \sinh\beta L)d_L$$

$$a_{12}(x,L) = (\sinh\beta x - \sin\beta x)(\sin\beta L - \sinh\beta L)d_L + (\cos\beta x - \cosh\beta x)(\cos\beta L + \cosh\beta L)d_L$$

$$a_{21}(x,L) = (\cosh\beta x - \cos\beta x)(\cos\beta L + \cosh\beta L)\beta d_L - (\sin\beta x + \sinh\beta x)(\sin\beta L + \sinh\beta L)\beta d_L$$

$$a_{22}(x,L) = (\cos\beta x - \cosh\beta x)(\sinh\beta L - \sin\beta L)\beta d_L + (\sin\beta x + \sinh\beta x)(\cos\beta L + \cosh\beta L)\beta d_L$$

$$d_L = (2 + 2\cos\beta L \cosh\beta L)^{-1}$$

$$\hat{f}_J = f_J/(\beta^3 EI)$$

$$\hat{m}_J = m_J/(\beta^3 EI)$$

$$\beta^4 = \omega^2 p/(EI)$$

The geometric compatibility and force equilibrium equations used for the synthesized system are as follows:

$$y^1(L) = y^2(H)$$

$$\theta^1(L) = -\theta^2(H)$$

$$\hat{m}_1 = \hat{m}_2$$

$$\hat{f}_1 = -\hat{f}_2$$

Combining the preceding equations and simplifying

$$\begin{bmatrix} a_{11}(L,L) + a_{11}(H,H) & a_{12}(L,L) - a_{12}(H,H) \\ a_{21}(L,L) - a_{21}(H,H) & a_{22}(L,L) + a_{22}(H,H) \end{bmatrix} \begin{bmatrix} \hat{f}_1 \\ \hat{m}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

Eq. (13) is in the form of

$$A\bar{f} = 0$$

The eigenvalue β_n of the system is obtained from

$$\det | A | = 0$$

The modal force vector $\bar{f}$ can then be found from Eq. (13). The mode shape of the system is found by substituting $\bar{f}$ into the response equation, Eq. (12).

For the purpose of comparison, let $L = H$; the eigenvalue equation is simplified to

$$\cosh 2\beta L \cos 2\beta L = 1$$

This solution is compared to the closed-form solution, and they are identical.

Conclusion

A method for substructure modal synthesis has been presented to calculate the eigensolution of a synthesized system. This method first determines the Modal Force vector of the system, then calculates the eigenvectors of the system.

1) The order of the Modal Force matrix equation is much smaller than the system dynamic matrix equation. This results in a large reduction in the number of numerical computations through the avoidance of inversions of large matrices.

2) The Modal Force vectors are derived during the eigensolution process, and thus they are available for dynamic stress calculations.

3) The eigensolution of the system in any frequency range can be obtained without prior elimination of the lower or higher frequency modes.

4) Information on substructure modes can be stored in the computer, allowing for different connections to form new structure configurations.

5) This is a direct method that can give exact results.

Appendix

If a system has a natural frequency identical to one of its substructures, the response equation of the substructure for the response function cannot be used at that particular frequency to derive the system mode shape. Instead, a modified approach is used.

For a substructure I that has its rth natural frequency identical to one of the system natural frequencies, the substructure has the following properties.

Property 1

All of the forces applied to the substructure I must be zero:

$$f_{IJ} = 0 \qquad J = 1,2,\ldots,s \tag{A1}$$

where s is the number of substructures in the system.

Property 2

Response vector x^I of substructure I must be proportional to its own rth modal vector:

$$x^I = \alpha\phi_r^I \tag{A2}$$

where α is a proportional constant and ϕ_r^I is the rth modal vector of substructure I. With properties 1 and 2, the modal force equation can be modified accordingly.

The three-substructure system discussed previously is used to demonstrate the modified approach. It is assumed that the substructure 1 has its rth natural frequency identical to the system natural frequency; the $f_{12} = 0$ is valid because of property 1. The modal force equation becomes

$$[H_{33}^2(\omega_r) + H_{11}^3(\omega_r)]\ [f_{23}] = [0] \tag{A3}$$

From Eq. (A3), vector f_{23} can be found. IN this case, f_{23} is a scalar, and it can be chosen as 1. With all of the modal force vectors known, the modal subvectors x^2 and x^3 can be obtained from the frequency response transfer equation.

Since ϕ_r^I is already known from substructure 1, the subvector x^1 can be obtained from property 2, Eq. (A2), as follows:

$$x^1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \alpha \phi_r^1 \tag{A4}$$

The condition $x_{12} = x_{21}$ determines α. With α determined, x^1 is obtained. After x^1, x^2, x^3 are found, the system modal vector for the rth mode is obtained by rearranging the known subvector x^1, x^2, x^3.

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